arXiv:gr-qc/0006103v1 29 Jun 2000

A first analysis regarding matter-dynamical diffeomorphism coupling [1]

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Abstract

A first attempt at adding *matter* degrees of freedom to the two-dimensional "vacuum" gravity model presented in [1] is analyzed in this paper. Just as in the previous pure gravity case, quantum diffeomorphism operators (constructed from a Virasoro algebra) possess a dynamical content; their gauge nature is recovered only after the classical limit. Emphasis is placed on the new physical modes modelled on a $SU(1,1)$-Kac-Moody algebra. The non-trivial coupling to "gravity" is a consequence of the natural semi-direct structure of the entire extended algebra. A representation associated with the discrete series of the rigid $SU(1,1)$ is revisited in the light of previously neglected crucial global features which imply the appearance of an $SU(1,1)$-Kac-Moody fusion rule, determining the rather entangled quantum structure of the physical system. In the classical limit, an action which explicitly couples gravity and matter modes governs the dynamics.

1 Introduction

A model for dealing with the potential dynamical content of the operators associated with diffeomorphisms in the quantum regime was recently presented in [1]. Accepting the (centrally-extended) abstract Virasoro group as the only physical input of the theory, a spacetime notion inside the group was found for only a critical combination of the central extension parameters. The resulting physical model presented an ensemble of coexisting spacetimes, mixed at the quantum level by the action of diffeomorphisms, while a correction to Polyakov's two-dimensional gravity action was found in the semi-classical limit.

[1]Work partially supported by the DGICYT.

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The main goal of this paper is the study of the coupling of new dynamical degrees of freedom, eventually interpretable as *matter*, to the previous model. Symmetry being our unique physical guide, we can resort only to algebraic structures in order to gain intuition regarding the manner of inserting the new modes. In particular, we are interested in a Lie algebra structure containing the Virasoro algebra as a subalgebra, in order to incorporate the *pure gravity* model, and coupling the diffeomorphism algebra to the new modes in a non-trivial way. It is by no means obvious that the enlarging the original algebra preserves the *critical* way in which spacetime emerged; that is, as associated with a critical value of the conformal charge.

The abstract approach followed here has the advantage of a clear and non-ambiguous definition of the mathematical structure of the physical system, but poses the serious problem of its physical interpretation. We have no other alternative than offering an *a posteriori* interpretation of the model depending of the concrete realization of the system. In some sense, any fundamental theory (as opposed to an effective one) must confront and solve this pitfall.

The simplest Lie algebra that fulfils the previous requirements is an affine Kac-Moody Lie algebra under the semi-direct action of the Virasoro group, and this will be our choice in the present paper. More specifically, and motivated by Julia and Nicolai's analysis [2] of two-dimensional gravity, where matter fields live on the quotient space of a finite-dimensional non-compact group, we shall choose a non-compact Kac-Moody group to implement the new degrees of freedom. Again on behalf of simplicity in this first approach to the problem, our choice will be the most manageable one; that is, we study $SU(1,1)$-Kac-Moody. This symmetry has also recently been considered for constructing coset models for black holes [3]. Thus, the Lie algebra $\mathcal{G}$ defining our physical system is:

$$
\begin{aligned}
[\hat{Z}_n, \hat{Z}^*{}_m] &= -2i\hat{\Phi}_{n+m} - 2(\alpha n + \frac{K}{2})\delta_{n+m}\hat{I} \\
\left[\hat{\Phi}_n, \hat{Z}_m\right] &= i\hat{Z}_{n+m}\ ,\ \left[\hat{\Phi}_n, \hat{Z}^*{}_m\right] = -i\hat{Z}^*{}_{n+m} \\
\left[\hat{\Phi}_n, \hat{\Phi}_m\right] &= \alpha n\delta_{n+m}\hat{I} \\
\left[\hat{L}_n, \hat{Z}_m\right] &= -m\hat{Z}_{n+m}\ ,\ \left[\hat{L}_n, \hat{Z}^*{}_m\right] = -m\hat{Z}^*{}_{n+m} \\
\left[\hat{L}_n, \hat{\Phi}_m\right] &= -m\hat{\Phi}_{n+m} + im\frac{K}{2}\delta_{n+m}\hat{I} \\
\left[\hat{L}_n, \hat{L}_m\right] &= (n-m)\hat{L}_{n+m} + \frac{1}{12}(cn^3 - c'n)\delta_{n+m}\hat{I} \\
\left[\hat{I}, ...\right] &= 0
\end{aligned}
\tag{1}
$$

The mathematical formalism to be used to derive the physical system from the previous algebra is the Group Approach to Quantization (GAQ). A non-technical presentation of it can be found in [1] (see references therein for deeper and more rigorous explanations). Here we just outline the basic features, which will be treated with more detail as they appear in the main text.

Our first step will be to consider the possible central extensions and pseudo-extensions[4] of the algebra, which are determined by its cohomology and pseudo-cohomology, respectively. This allows a distinction (if no anomalous reduction is present at the representation level) between the dynamical (or basic) degrees of freedom and the kinematical ones. The first set define the physical phase space of the system, whereas the second one induces evolution flows along the former. In (1) the cohomology has already been taken into account.

After cohomology is analysed, the next crucial point is to *exponentiate* the Lie algebra $\mathcal{G}$ to a Lie group G. Dynamics are contained in the group rather than in the algebra, and global questions, which become visible at the group level but not at the algebra one, prove to be critical in some points of our analysis. In this sense, we shall see in **subsection 3.2** how imposing of globallity on the wave functions brings about the presence of restrictions in the representation that guarantee the consistency (unitarity) of the theory. Once we have a centrally-extended group law, we can compute the left- and right-invariant vector fields as well as the quantization one-form Θ (that is, the component of the canonical left-invariant one-form which is dual to the central vector field $\hat{I}$). The kinematical vector fields, those comprising the kernel of the Lie-algebra two-cocycle, can be characterised as spanning the left-invariant characteristic subalgebra $\mathcal{G}_\Theta = Ker(\Theta) \cap Ker(d\Theta)$. With these elements at hand, one can develop a semi-classical formalism, or a directly quantum one.

The main element that defines the semi-classical formalism is the classical phase space $\mathcal{M}$, obtained by taking the quotient of the group by the equations of motion as well as by the $U(1)$ central extension subgroup ($\mathcal{M} = G/(\mathcal{G}_\Theta \otimes U(1))$. It can be parametrized in terms of the basic Noether invariants, $i_{X_i^R}\Theta$, where the respective X_i^L are not present in the characteristic subalgebra. The symplectic form is defined by $d\Theta$, which properly falls down to the quotient. Finally, an action S can be derived by integrating Θ over trajectories on the group: $S = \int \Theta$. This formalism is the one used at the classical level.

The quantum theory is obtained after reducing the regular representation, defined by the action of the right-invariant vector fields over the complex $U(1)$-functions with support on the group, by means of a polarization (see **subsection 3.1**) constructed by using left-invariant vector fields. These latter commute with the right-invariant ones, thus respecting the group action. The resulting representation must be unitary in order to introduce a standard quantum probability notion.

The organization of the paper parallels that of [1] and is as follows. In **section 2** we present the elements of the semi-classical formalism, emphasizing those appearing in the classical limit. **Section 3** deals with the quantum theory and is divided into three subsections which deal with reducibility, unitarity, and a possible contraction phenomenon in the theory, respectively. With the previous elements in mind, **section 4** tries to give a physical interpretation to the mathematical objects introduced in the preceding sections. Finally, several comments and reflections are presented in **section 5**.

Before we start, let us insist on the fact that the reasoning and results in the following

[4] Redefinitions of certain generators with non-trivial dynamical consequences. They are crucial for semi-simple groups, where true cohomology is trivial.

sections must be contemplated in the spirit and context of [1].

2 Semi-classical formalism

As explained in the previous section, our first step in the construction of the model is the *exponentiation* to a group law from the Lie algebra (1), which defines the physical system, since physics are encoded in the (global) Lie group structure.

To accomplish this goal, we use the $SU(1,1)$ local group law:

$$\begin{aligned} z'' &= z\eta'^{-2} + \kappa z' + \frac{2z'}{1+\kappa'}[z^{*\prime} z\eta'^{-2} + z^* z'\eta^2] \\ z^{*\prime\prime} &= z^*\eta'^2 + \kappa z^{*\prime} + \frac{2z^{*\prime}}{1+\kappa'}[z' z^*\eta'^2 + zz^{*\prime}\eta^{-2}] \\ \eta'' &= \sqrt{\frac{2}{1+\kappa''}}\left[\sqrt{\frac{1+\kappa}{2}}\sqrt{\frac{1+\kappa'}{2}}\eta\eta' + \sqrt{\frac{2}{1+\kappa}}\sqrt{\frac{2}{1+\kappa'}}z^{*\prime} z\eta^{*\prime}\eta\right], \end{aligned} \tag{2}$$

where

$$\begin{aligned} \kappa &= \sqrt{1+2zz^*} \;\;;\;\; \kappa'' = \kappa\kappa' + zz^{*\prime}\eta'^{-2} + z^* z'\eta'^2 \\ \eta &= e^{i\phi} \end{aligned}$$

The law for the Virasoro subgroup can be found in [1] and references therein. Expanding the elements of $SU(1,1)$ in a formal Laurent series to conform to the loop group law [5], and using the techniques developed in [4] to construct the cocycles for the central extensions as well as the (semi-direct) action of the Virasoro subgroup on the $SU(1,1)$-Kac-Moody subgroup, we find the following group law:

$$\begin{aligned} z^{n\prime\prime} &= z^n + A^n_k(l) z^{k\prime} - iA^m_k(l) z^{n-m}\phi^k + A^m_k(l)A^p_l(l)(z^{n-m-p}z^{k\prime}z^{*l\prime} + z^{*n-m-p}z^{k\prime}z^{l\prime} \\ &- \frac{1}{2}z^{n-m-p}\phi^{k\prime}\phi^{l\prime}) + 2A^p_l(l)z^{n-m-p}z^{*m}z^{l\prime} + ... \\ z^{*n\prime\prime} &= z^{*n} + A^n_k(l) z^{*k\prime} + iA^m_k(l) z^{*n-m}\phi^k + A^m_k(l)A^p_l(l)(z^{*n-m-p}z^{*k\prime}z^{l\prime} + z^{n-m-p}z^{*k\prime}z^{*l\prime} \\ &- \frac{1}{2}z^{*n-m-p}\phi^{k\prime}\phi^{l\prime}) + 2A^p_l(l)z^{*n-m-p}z^{m}z^{*l\prime} + ... \\ \phi^{n\prime\prime} &= \phi^n + A^n_k(l)\phi^k - iA^m_k(l)(z^{n-m}z^{*k\prime} - z^{*n-m}z^{k\prime}) - A^m_k(l)A^p_l(l)(z^{n-m-p}z^{*k\prime} \\ &+ z^{*n-m-p}z^{k\prime})\phi^l + ... \\ l^{\prime\prime m} &= l^m + l^{\prime m} + ipl^{\prime p}l^{m-p} + \frac{(ip)^2}{2!}l^{\prime p}l^n l^{m-n-p} + ... + \sum_{n^1+...+n^j+p=m}\frac{(ip)^j}{j!}l^{\prime p}l^{n^1}...l^{n^j} + ... \\ \varphi'' &= \varphi + \varphi' + \frac{K}{2}\xi_{cobKM} + \alpha\xi_{KM} + \frac{c}{24}\xi_{Vir} - \frac{c'}{24}\xi_{cobVir} \end{aligned} \tag{3}$$

[5] We expand up to the third order in the parameters, which is sufficient for our purposes.

where

$$
\begin{aligned}
A_n^k(l) &= \delta_n^k + (in)l^{k-n} + \sum_{r=2}\sum_{n+m_1+...+m_r=k} \frac{(in)^r}{r!} l^{m_1}...l^{m_r} \\
\xi_{KM} &= \frac{-1}{2}[inA_k^{-n}(l)(2z^n z^{*k\prime} + 2z^{*n}z^{k\prime} - \phi^n\phi^{k\prime}) - iA_k^m(l)A_k^{-n-m}(l)i(n-m)(z^n z^{*k\prime}\phi^{l\prime} \\
&- z^{*n}z^{k\prime}\phi^{l\prime}) - iA_k^{-n-m}(l)in(z^{*m}\phi^n z^{k\prime} - z^m\phi^n z^{*k\prime})] + ... \qquad (4) \\
\xi_{Vir} &= -[(-i)(-n)n^2 l^{-n\prime}l^n + \frac{(-i)^2}{2!}n_1n_2(n_1+n_2)^2 l^{n_1\prime}l^{n_2\prime}l^{-n_1-n_2} \\
&- \frac{i^2}{2!}(n_1+n_2)^2(n_1^2+n_2^2+n_1n_2)l^{-n_1-n_2\prime}l^{n_1}l^{n_2} + ...] \\
\xi_{CobKM} &= \phi^{0\prime\prime} - \phi^{0\prime} - \phi^0 \\
\xi_{CobVir} &= l^{0\prime\prime} - l^{0\prime} - l^0.
\end{aligned}
$$

With the explicit group law at our disposal, the left- and right-invariant vector fields can be systematically computed, obtaining:

$$
\begin{aligned}
\tilde{X}^L_{z^r} &= \frac{\partial}{\partial z^r} - i\phi^{n-r}\frac{\partial}{\partial z^n} + z^{n-p-r}z^{*p}\frac{\partial}{\partial z^n} - \frac{1}{2}\phi^{n-p-r}\phi^p\frac{\partial}{\partial z^n} + z^{*n-p-r}z^{*p}\frac{\partial}{\partial z^{*n}} - iz^{*n-r}\frac{\partial}{\partial \phi^n} \\
&- z^{*n-p-r}\phi^p\frac{\partial}{\partial\phi^n} + ... + [-i(\frac{K}{2}+\alpha r)z^{*-r} - (\frac{K}{2}+\frac{\alpha}{2}(r-m))z^{*m}\phi^{-m-r} + ...]\frac{\partial}{\partial\varphi} \\
\tilde{X}^L_{z^{*r}} &= \frac{\partial}{\partial z^{*r}} + i\phi^{n-r}\frac{\partial}{\partial z^{*n}} + z^{*n-p-r}z^p\frac{\partial}{\partial z^{*n}} - \frac{1}{2}\phi^{n-p-r}\phi^p\frac{\partial}{\partial z^{*n}} + z^{n-p-r}z^p\frac{\partial}{\partial z^{*n}} + iz^{n-r}\frac{\partial}{\partial\phi^n} \\
&- z^{n-p-r}\phi^p\frac{\partial}{\partial\phi^n} + ... + [i(\frac{K}{2}-\alpha r)z^{-r} - (\frac{K}{2}-\frac{\alpha}{2}(r-m))z^m\phi^{-m-r} + ...]\frac{\partial}{\partial\varphi} \qquad (5) \\
\tilde{X}^L_{\phi^r} &= \frac{\partial}{\partial\phi^r} + (\frac{\alpha}{2}ir\phi^{-r} + ...)\frac{\partial}{\partial\varphi} \\
\tilde{X}^L_{l^r} &= \tilde{X}^{L\,Vir}_{l^r} + i(n-r)z^{n-r}\frac{\partial}{\partial z^n} + i(n-r)z^{*n-r}\frac{\partial}{\partial z^{*n}} + i(n-r)\phi^{n-r}\frac{\partial}{\partial\phi^n} \\
\Xi &= \tilde{X}^L_\varphi = \frac{\partial}{\partial\varphi}
\end{aligned}
$$

for the left-invariant vector fields, and

$$
\begin{aligned}
\tilde{X}^R_{z^r} &= A_r^n(l)\frac{\partial}{\partial z^n} + 2A_r^p(l)z^{n-m-p}z^{*m}\frac{\partial}{\partial z^n} + iA_r^m(l)z^{*n-m}\frac{\partial}{\partial\phi^n} + ... \\
&+ [i(\frac{K}{2}-\alpha n)A_r^{-n}(l)z^{*n} - \frac{\alpha}{2}nA_r^{-n-m}(l)z^{*m}\phi^n + ...]\frac{\partial}{\partial\varphi} \\
\tilde{X}^R_{z^{*r}} &= A_r^n(l)\frac{\partial}{\partial z^{*n}} + 2A_r^p(l)z^{*n-m-p}z^m\frac{\partial}{\partial z^{*n}} - iA_r^m(l)z^{n-m}\frac{\partial}{\partial\phi^n} + ... \\
&+ [-i(\frac{K}{2}+\alpha n)A_r^{-n}(l)z^n + \frac{\alpha}{2}nA_r^{-n-m}(l)z^m\phi^n + ...]\frac{\partial}{\partial\varphi} \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{X}^R_{\phi^r} &= A^n_r(l)\frac{\partial}{\partial\phi^n} - iA^m_r(l)z^{n-m}\frac{\partial}{\partial z^n} + iA^m_r(l)z^{*n-m}\frac{\partial}{\partial z^{*n}} + ... + (\frac{\alpha}{2}nA^{-n}_r(l)\phi^n + ...)\frac{\partial}{\partial\varphi} \\
\tilde{X}^R_{l^r} &= \tilde{X}^{R\,Vir}_{l^r} = \frac{\partial}{\partial l} + irl^{m-r}\frac{\partial}{\partial l^m} + [\frac{ic}{24}r^3 l^{-r} - \frac{c}{24}r^2 \sum_{n_1+n_2=-r}(n_1^2+n_2^2+n_1n_2)l^{n_1}l^{n_2} \\
&- \frac{ic'}{24}rl^{-r} + \frac{c'}{24}\sum_{n_1+n_2=-r}\frac{r^2}{2}l^{n_1}l^{n_2} + ...]\frac{\partial}{\partial\varphi} \\
\Xi &= \tilde{X}^R_\varphi = \frac{\partial}{\partial\varphi} \quad ,
\end{aligned}
$$

where $\tilde{X}^{LVir}_{l^r}$ and $\tilde{X}^{R\,Vir}_{l^r}$ are the expressions that can be found in [1].

In order to obtain the quantization one-form Θ, that is, the vertical component of the canonical left-invariant one-form on the group, we impose duality on the left-invariant vector fields ($\Theta(\tilde{X}^L_{z^r}) = \Theta(\tilde{X}^L_{z^{*r}}) = \Theta(\tilde{X}^L_{\phi^r}) = \Theta(\tilde{X}^L_{l^r}) = 0$ and $\Theta(\Xi) = 1$). The resulting expression is:

$$
\begin{aligned}
\Theta &= \Theta^{KM} + \Theta^{Vir} + \Theta^{Int} + d\varphi \\
\Theta^{KM} &= \frac{-i\alpha}{2}r\phi^{-r}d\phi^r + [i(\frac{K}{2}+\alpha r)z^{*-r} + \alpha(n+r)z^{*n}\phi^{-n-r} + ...]dz^r + [i(\frac{-K}{2}+\alpha r)z^{-r} \\
&- \alpha(n+r)z^n\phi^{n-r} + ...]dz^{*r} \\
\Theta^{Vir} &= \frac{i}{24}(cn^2 - c')nl^{-n}dl^n + \qquad (7) \\
&+ \sum_{\substack{k=2 \\ n_1+...n_k=-n}} \frac{(-i)^k}{24}[cn_1^2 - c' + cn^2\sum_{m=2}^{k}\frac{1}{m!}]n_1...n_k l^{n_1}...l^{n_k}dl^n \\
\Theta^{Int} &= \sum_{\substack{j=0 \\ n_1+...+n_j+k=-r}} (-i)^j f^k(z^r, z^{*r}, \phi^r)n_1...n_j dl^r \quad ,
\end{aligned}
$$

where

$$
\begin{aligned}
f^k(z^r, z^{*r}, \phi^r) &= -i\alpha(n+k)nz^{-n+k}z^{*n} - \frac{iK}{2}(n+k)z^{n+k}z^{*-n} - i\alpha(n+k)nz^{*-n+k}z^n \\
&+ \frac{iK}{2}(n+k)z^{*n+k}z^{-n} + \frac{i\alpha}{2}n(n+k)\phi^{-n}\phi^{n+k} + \alpha nmz^n\phi^m z^{*k-n-m} \qquad (8) \\
&- \alpha nmz^{*n}\phi^m z^{k-n-m} + ... \quad .
\end{aligned}
$$

For both the classical and the quantum theory, the identification of the characteristic subalgebra $\mathcal{G}_\Theta$ ($= Ker\Theta \cap Kerd\Theta$), is a crucial point. The central extension structure simplifies this search, reducing it to the determination of the kernel of the Lie-algebra cocycle. Considering the commutation relationships (1) we notice that the composition of $\mathcal{G}_\Theta$ depends of the actual values of the central extensions c and α, as well as pseudo-extensions c' and K, giving rise to a wide range of possibilities. We shall study the specific choice of the extension parameters that fits our physical purposes. Following [1], we want

to find a $sl(2,\mathbb{R})$ (Virasoro-)subalgebra inside the characteristic subalgebra in order to construct a spacetime notion. This can be achieved if we impose $c = c' - 3\frac{K^2}{\alpha}$ [6]. In fact, the linear combination, $\tilde{\tilde{X}}^L_{l^i} = \tilde{X}^L_{l^i} + \frac{K}{2\alpha}\tilde{X}^L_{\phi^i}$ ($i \in \{-1,0,1\}$), then enters $\mathcal{G}_\Theta$. Furthermore, we require $\frac{K}{2\alpha} \notin \mathbb{Z}$ in order not to lose dynamical modes in the physical (matter) fields. Then we have:

$$\mathcal{G}_\Theta = \langle \tilde{X}^L_{\phi^0}, \tilde{\tilde{X}}^L_{l^{-1}}, \tilde{\tilde{X}}^L_{l^0}, \tilde{\tilde{X}}^L_{l^1} \rangle \quad . \tag{9}$$

The linear combination giving rise to the $sl(2,\mathbb{R})$ inside $\mathcal{G}_\Theta$ suggests the possibility of generalizing it for the rest of the Virasoro modes, closing again a Virasoro subalgebra ($\overline{Vir}$):

$$\tilde{X}^L_{l^n} \to \tilde{\tilde{X}}^L_{l^n} = \tilde{X}^L_{l^n} + \frac{K}{2\alpha}\tilde{X}^L_{\phi^n} \quad , \quad \forall n \ . \tag{10}$$

In that case, the pure Kac-Moody commutation relations remain the same, but the Virasoro ones take the form:

$$\begin{aligned}
\left[\tilde{\tilde{X}}^L_{l^n}, \tilde{X}^L_{z_m}\right] &= -i(m + \frac{K}{2\alpha})\tilde{X}^L_{z_{n+m}} \ , \ \left[\tilde{\tilde{X}}^L_{l^n}, \tilde{X}^L_{z^*_m}\right] = -i(m - \frac{K}{2\alpha})\tilde{X}^L_{z^*_{n+m}} \\
\left[\tilde{\tilde{X}}^L_{l^n}, \tilde{X}^L_{\phi_m}\right] &= -im\tilde{X}^L_{\phi_{n+m}} \\
\left[\tilde{\tilde{X}}^L_{l^n}, \tilde{\tilde{X}}^L_{l^m}\right] &= i(n-m)\tilde{\tilde{X}}^L_{l^{n+m}} + \frac{i}{12}(cn^3 - (c' - 3\frac{K^2}{\alpha})n)\delta_{n+m}\Xi \quad .
\end{aligned} \tag{11}$$

This basis will be more suited for the polarization analysis in the next section.

The classical equations of motion for the model consist of the dynamical system defined in terms of the vector fields in $\mathcal{G}_\Theta$, which dictate the evolution of the parameters in the group. Looking at the explicit form of the left-invariant vector fields, we observe that the group parameters are constant under $\tilde{X}^L_{\phi^i}$, so we can ignore them in the equations of motion. Thus the evolution is parametrized solely by the *space-time* $SL(2,\mathbb{R})$ subgroup:

$$\frac{\partial g^n_i}{\partial \tilde{\lambda}_0} = (\tilde{X}^L_{l^0})g^n_i \quad , \quad \frac{\partial g^n_i}{\partial \tilde{\lambda}_1} = (\tilde{X}^L_{l^1})g^n_i \quad , \quad \frac{\partial g^n_i}{\partial \tilde{\lambda}_{-1}} = (\tilde{X}^L_{l^{-1}})g^n_i \quad , \quad i \in \{1,2,3,4\} \ , \tag{12}$$

where $g^n_1 = z^n, g^n_2 = z^{*n}, g^n_3 = \phi^n, g^n_4 = l^n$; while $\tilde{\lambda}_0, \tilde{\lambda}_1$ and $\tilde{\lambda}_{-1}$ are the parameters of the vector fields $\tilde{X}^L_{l^0}, \tilde{X}^L_{l^1}$ and $\tilde{X}^L_{l^{-1}}$, respectively.

The following explicit equations of motion are *exact*, in spite of the development of the Kac-Moody subgroup up to the third order.

$$\begin{aligned}
\frac{\partial g^m_i}{\partial \tilde{\lambda}_0} &= img^m_i \ , \ \frac{\partial g^m_i}{\partial \tilde{\lambda}_1} = i(m-1)g^{m-1}_i \ , \ \frac{\partial g^m_i}{\partial \tilde{\lambda}_{-1}} = i(m+1)g^{m+1}_i \ , \ i \in \{1,2,3\} \\
\frac{\partial l^m}{\partial \tilde{\lambda}_0} &= iml^m \quad \text{for} \quad m \neq 0 \quad , \quad \frac{\partial l^0}{\partial \tilde{\lambda}_0} = 1
\end{aligned}$$

[6] We shall see in the next section that quantum theory imposes a correction to this relationship.

$$
\begin{aligned}
\frac{\partial l^m}{\partial\tilde{\lambda}_1} &= i(m-1)l^{m-1} \quad \text{for} \quad m\neq 1 \quad , \quad \frac{\partial l^1}{\partial\tilde{\lambda}_1}=1 \\
\frac{\partial l^m}{\partial\tilde{\lambda}_{-1}} &= i(m+1)l^{m+1} \quad \text{for} \quad m\neq -1 \quad , \quad \frac{\partial l^{-1}}{\partial\tilde{\lambda}_{-1}}=1 \ .
\end{aligned} \tag{13}
$$

These equations have the same form as those found in [1] (in fact, they are exactly the same for the l^n), so they can be solved exactly and present the structure:

$$
\begin{aligned}
g_i^n(\lambda_{-1},\lambda_0,\lambda_1) &= g_i^n(\lambda_{-1},\lambda_1)e^{in\lambda_0} \quad , \quad i\in\{1,2,3\} \\
l^n(\lambda_{-1},\lambda_0,\lambda_1) &= l^n(\lambda_{-1},\lambda_1)e^{in\lambda_0} \ , \ n\neq 0 \ \ ; \ \ l^0=\lambda_0 \ \ ,
\end{aligned} \tag{14}
$$

where $\tilde{\lambda}_0=\lambda_0$, $\tilde{\lambda}_1=\lambda_1 e^{i\lambda_0}$, $\tilde{\lambda}_{-1}=\lambda_{-1}e^{-i\lambda_0}$.

The symplectic manifold characterizing the classical physical system is obtained by taking the quotient of the (non-extended) group by the equations of motion. The symplectic form is defined by $d\Theta$, which passes to the quotient. This phase space can be parametrized by the Noether invariants of the group parameters whose left-invariant vector fields are not present in $\mathcal{G}_\Theta$. This is the reason for referring to these modes as dynamical (or basic) degrees of freedom. We give the explicit expressions for the Noether invariants up to the second order:

$$
\begin{aligned}
Z_r &= i_{\tilde{X}^R_{z^r}}\Theta = 2i(\frac{K}{2}+\alpha r)z^{*-r}+2\alpha(m+r)z^{*m}\phi^{-m-r}-2r(\frac{K}{2}+\alpha n)l^{n-r}z^{-n}+... \\
{Z^*}_r &= i_{\tilde{X}^R_{z^{*r}}}\Theta = -2i(\frac{K}{2}-\alpha r)z^{-r}-2\alpha(m+r)z^{m}\phi^{-m-r}-2r(\frac{K}{2}-\alpha n)l^{n-r}z^{*-n}+... \\
\Phi_r &= i^R_{\tilde{X}_{\phi^r}}\Theta = -i\alpha r\phi^{-r}+z^{m-r}z^{*-m}[(\frac{K}{2}+\alpha m)-(\frac{-K}{2}+\alpha(r-m))]+\alpha n r\phi^{-n}l^{n-r}+... \\
L_r &= i^R_{\tilde{X}_{l^r}}\Theta = L_r^{Vir}+\frac{i\alpha}{2}n(-n-r)[\phi^n\phi^{-n-r}-4z^{-n-r}z^{*n}]-\frac{Ki}{2}(2n-r)z^{n-r}z^{*-n}+... \ ,
\end{aligned} \tag{15}
$$

where, again, the superscript Vir denotes the object that can be found in [1].

The configuration-like description of the system, better suited for a Lagrangian formalism, can be obtained after defining the fields:

$$
\begin{aligned}
F_z(\lambda_{-1},\lambda_0,\lambda_1) &\equiv \sum_n z^n(\lambda_{-1},\lambda_0,\lambda_1)=\sum_n z^n(\lambda_{-1},\lambda_1)e^{in\lambda_0} \\
F_{z^*}(\lambda_{-1},\lambda_0,\lambda_1) &\equiv \sum_n z^{*n}(\lambda_{-1},\lambda_0,\lambda_1)=\sum_n z^{*n}(\lambda_{-1},\lambda_1)e^{in\lambda_0} \\
F_\phi(\lambda_{-1},\lambda_0,\lambda_1) &\equiv \sum_n \phi^n(\lambda_{-1},\lambda_0,\lambda_1)=\sum_n \phi^n(\lambda_{-1},\lambda_1)e^{in\lambda_0} \\
F_l(\lambda_{-1},\lambda_0,\lambda_1) &\equiv \sum_n l^n(\lambda_{-1},\lambda_0,\lambda_1)=\lambda_0+\sum_{n\neq 0} l^n(\lambda_{-1},\lambda_1)e^{in\lambda_0} \quad ,
\end{aligned} \tag{16}
$$

where the explicit solution to the classical equations of motion has been used in the second equality.

We can take the classical limit $c \to \infty$, that is, $R \to \infty$ (exactly in the same way as we did in [1]), after we have made the linear change of variables $u = \frac{1}{2}(\lambda_1 + \lambda_{-1}), v = \frac{1}{2}(\lambda_1 - \lambda_{-1}), \lambda = \lambda_0$ and imposed the Casimir constraint, which compels the previously defined fields to *live* on AdS spacetime. In this situation we find:

$$\begin{aligned} F_{g_i}(u,\lambda) &\equiv F^{R\to\infty}_{g^i\,AdS}(u,\lambda) = \sum_n g_i{}^n(u) e^{in\lambda} \quad , \; i \in \{1,2,3\} \\ F_{l^n}(u,\lambda) &\equiv F^{R\to\infty}_{l\,AdS}(u,\lambda) = \lambda + \sum_{n\neq 0} l^n(u) e^{in\lambda} \quad . \end{aligned} \tag{17}$$

In this limit, it is easy to express the g_i^n's in terms of the $F_{g_i}(u,\lambda)$ and we can write Θ in the configuration-like variables. By defining the action as $S = \int \Theta$, we encounter:

$$S = S^{KM} + S^{Vir} + S^{Int} \quad ,$$

where S^{KM} is the action for pure $SU(1,1)$-Kac-Moody [7], S^{Vir} is the corrected Polyakov action for $2D$ quantum gravity found in [1], and S^{Int} is an interaction term, which couples the gravitational degrees of freedom coming from the Virasoro algebra to the new dynamical modes of Kac-Moody. The structure of this term is:

$$S^{Int} = \int du d\lambda \frac{\mathcal{F}(F_z, F_{z^*}, F_\phi)\partial_u F_l}{\partial_\lambda F_l} \quad , \tag{18}$$

where $\mathcal{F}(F_z, F_{z^*}, F_\phi)$ is a functional of the fields F_z, F_{z^*}, F_ϕ, whose actual form (obtained in a perturbative way) is not relevant here.

We finally point out the natural appearance of an interaction term between the gravitational degrees of freedom and the new ones.

3 Quantum model

3.1 Reduction

In a group approach to quantum theory, the quantum realization of the physical model is accomplished by the construction of an irreducible and unitary representation of the chosen physical group. This representation is obtained from the regular representation, which is highly reducible, by imposing certain conditions to the wave functions. These conditions are encoded in the so-called polarization subalgebra $\mathcal{P}$, which is a left-invariant maximal horizontal subalgebra including the characteristic subalgebra. That is, it includes $\mathcal{G}_\Theta$ and one mode for each conjugated pair of the dynamical degrees of freedom.

[7]Here only evaluated in a perturbative manner up to the second order. Since we shall not make use of the Lagrangian formalism, we do not give the explicit expression here. We only notice that at first order the Kac-Moody modes behave as free scalar fields, which become coupled at higher orders.

Before we proceed to the explicit construction, we redefine the generators of the algebra in order to recover exactly the commutators (1):

$$
\begin{aligned}
\hat{L}_n &\equiv i\tilde{X}^R_{l^n}\ ,\ \hat{I} \equiv i\Xi\ ,\ \hat{G}^i_n \equiv \tilde{X}^R_{g^n_i} \\
\text{and}\quad \hat{G}^1_n &\equiv \hat{Z}_n\ ,\ \hat{G}^2_n \equiv \hat{Z}^*{}_n\ ,\ \hat{G}^3_n \equiv \hat{\Phi}_n\ .
\end{aligned}
\tag{19}
$$

A characteristic feature of infinite-dimensional groups is the possibility of the appearance of non-equivalent polarizations. Different polarizations lead to physically different systems, that is, the dynamics are not equivalent. In fact, in our case there are two non-equivalent polarizations due to the presence of the Kac-Moody subgroup. Let us focus ourselves on this Kac-Moody group and ignore the Virasoro subgroup for the time being. Considering the selected $\mathcal{G}_\Theta$ and looking at the commutation relations, we find two possibilities:

$$
\begin{aligned}
\mathcal{P}^N_{KM} &= \langle \tilde{X}^L_{\phi^{n\leq 0}}, \tilde{X}^L_{z^r} \rangle \\
\mathcal{P}^S_{KM} &= \langle \tilde{X}^L_{\phi^{n\leq 0}}, \tilde{X}^L_{z^{p\leq 0}}, \tilde{X}^L_{z^{*q<0}} \rangle \quad .
\end{aligned}
\tag{20}
$$

The first case, $\mathcal{P}^N_{KM}$, called *natural* polarization, is characterised by the presence of all the operators associated with the negative (or positive) roots of the semi-simple algebra, whereas in the second one, $\mathcal{P}^S_{KM}$, called *standard*, coexist operators associated with all the roots of the finite algebra (see [4] for details in $SU(2)$-Kac-Moody). In this paper we shall consider only the second case, since unitarity compels α to be zero in the natural polarization, which is too a severe condition and makes the dynamics less interesting in this framework, although a meaningful gravitational model can still be constructed with this natural polarization [5].

In a naïve way, we would expect that the inclusion of the Virasoro modes would not significatively alter the construction of a *standard-like* polarization associated with our selected $\mathcal{G}_\Theta$. The union of the polarization for the two well-studied separate cases (Kac-Moody and Virasoro), $\mathcal{P}^S = \mathcal{P}^S_{KM} \cup \mathcal{P}_{\overline{Vir}}$, seems to be a good candidate for a polarization of the entire group and, in fact, this would be the case if the only Virasoro mode in $\mathcal{G}_\Theta$ was $\tilde{X}^L_{l^0}$ or if the parameter K was zero. The first possibility must be rejected since our aim is to generalize the model in [1], for which the presence of a $sl(2,\mathbb{R})$, from the Virasoro algebra, inside $\mathcal{G}_\Theta$ is fundamental for the spacetime notion. The second one must also be discarded on unitarity grounds as will be seen below.

Unfortunately $\mathcal{P}^S_{KM}$ is not left invariant under the action of $\mathcal{P}_{\overline{Vir}}$. Indeed, the commutator $[\tilde{\tilde{X}}^L_{l^1}, \tilde{X}^L_{z^0}]$ is proportional to $\tilde{X}^L_{z^1}$ (since $K \neq 0$), which is absent from the subalgebra $\mathcal{P}^S_{KM}$. And what is worse, there is no full polarization containing the whole characteristic subalgebra [8]. This is an intrinsic, algebraic pathology that cannot be avoided.

We shall call this situation a $SL(2,\mathbb{R})-anomaly$, to distinguish it from the more usual case in conformal field theories (like WZW models or string theory) where the entire

[8]To be precise, a full polarization can be constructed for $\frac{K}{2\alpha} = -1$, but the resulting representation is not exponentiable, according to the *fusion rule* to be derived in the next section. The same globallity requirement excludes the *natural-like* full polarizations for $\alpha \neq 0$.

Virasoro algebra is devoid of dynamical content, but nevertheless cannot be included inside the polarization as a whole. The latter is conventionally referred to as conformal anomaly.

A well established procedure in the framework of group quantization, whenever the whole characteristic algebra cannot be included inside the polarization, is to correct the operators in $\mathcal{G}_\Theta$ with higher-order terms in the left enveloping algebra, giving rise to a higher-order characteristic algebra and/or polarization. The simplest physical example exhibiting this solution, either explicit or implicitly, is the case of the Schrödinger group [6] in non-linear quantum optics [7]. The symmetry of this problem is that of the harmonic oscillator group $(\hat{a}, \hat{a}^\dagger, \hat{H})$ with the two extra generators $\hat{l}_{-1}, \hat{l}_1$, closing with $\hat{H} \equiv \hat{l}_0$ a $SL(2,\mathbb{R})$ algebra, which constitutes the characteristic subalgebra. This symmetry is intrisicly anomalous, since no first-order full polarization including the $SL(2,\mathbb{R})$ can be found. Resorting to the left-enveloping algebra, an extra $SL(2,\mathbb{R})$ of the form (left version of) $\frac{(\hat{a})^2}{2}, \hat{a}\hat{a}^\dagger, \frac{(\hat{a}^\dagger)^2}{2}$ can be found, in such a way that the difference with the first-order one can be included in the polarization, thus solving the anomalous reduction problem.

Following the same reasoning in the present case, we seek new operators in the left-enveloping algebra. Inspired by the abovementioned example, as well as the WZW models, the left Sugawara operators in the Kac-Moody-quadratic enveloping algebra constitute a natural guess. The standard expressions of the left- and right-invariant forms of these generators (analogous to $\frac{(\hat{a})^2}{2}, \hat{a}\hat{a}^\dagger, \frac{(\hat{a}^\dagger)^2}{2}$), in terms of the Non-Pseudo-extended (NP) generators $X_{NP}{}^{L,R}_{g_i^n} \equiv \tilde{X}^{L,R}_{g_i^n} - \frac{K}{2}\delta_{n,0}\delta_{i,3}\Xi$, are:

$$(\tilde{X}^{Sug}_{l^n})^{L,R} \equiv \frac{1}{2\alpha} : \sum_m k^{ij} X_{NP}{}^{L,R}_{g_i^n} X_{NP}{}^{L,R}_{g_j^{n-m}} : \quad i,j \in \{1,2,3\} \quad , \tag{21}$$

where : : denotes standard normal ordering and k^{ij} is the Killing metricof the rigid group. Their commutation relations with the first-order left-invariant vectors are:

$$\begin{aligned} \left[(\tilde{X}^{Sug}_{l^n})^L, \tilde{X}^L_{g_i^m}\right] &= -im\tilde{X}^L_{g_i^{n+m}} \\ \left[(\tilde{X}^{Sug}_{l^n})^L, (\tilde{X}^{Sug}_{l^m})^L\right] &= i(n-m)(\tilde{X}^{Sug}_{l^{n+m}})^L + \frac{i}{12}c^{Sug}n^3\delta_{n+m}\Xi \\ \left[\tilde{X}^L_{l^n}, (\tilde{X}^{Sug}_{l^m})^L\right] &= i(n-m)(\tilde{X}^{Sug}_{l^{n+m}})^L + \frac{i}{12}c^{Sug}n^3\delta_{n+m}\Xi \quad . \end{aligned} \tag{22}$$

We see that the Sugawara operators close a Virasoro algebra with $c^{Sug} = \frac{\alpha dim(G)}{-\mathrm{g}+\alpha}$, g being the dual Coxeter number (see [8]). These commutators can be classically checked, and the central extensions then fixed by consistency with Jacobi identities. Notice that these higher-order operators close a proper algebra with the first-order ones.

These new generators are used to correct the first-order characteristic algebra and to define a higher-order polarization in terms of the difference of the first- and second-order Virasoro subalgebras: $(\tilde{X}^I_{l^n})^L = \tilde{X}^L_{l^n} - (\tilde{X}^{Sug}_{l^n})^L$ [9]. The commutation relations for the

[9] Strictly speaking, we just need to correct the $\tilde{X}^L_{l^1}$ generator, but the analysis is simpler if we extend this correction to the entire Virasoro subalgebra.

intrinsic Virasoro generators, $(\tilde{X}^I_{l^n})^L$, with themselves and the Kac-Moody modes are:

$$\begin{aligned} \left[(\tilde{X}^I_{l^n})^L, (\tilde{X}^I_{l^m})^L\right] &= i(n-m)(\tilde{X}^I_{l^{n+m}})^L + \frac{i}{12}[(c - c^{Sug})n^3 - c'n]\delta_{n+m}\hat{I} \\ \left[(\tilde{X}^I_{l^n})^L, \hat{X}_{g^m_i}\right] &= 0 \quad , \end{aligned} \tag{23}$$

and, therefore, the corrected Virasoro group does not display the fatal non-diagonal action on the Kac-Moody polarization. This detail allows us to construct a polarization as the simple union of the Kac-Moody and the intrinsic Virasoro one, permitting in particular the presence of the corrected characteristic subalgebra inside the polarization:

$$\mathcal{P} = \mathcal{P}^S_{KM} \cup \langle (\tilde{X}^I_{l^{n\leq 1}})^L \rangle \quad . \tag{24}$$

The irreducibility of the representation is guaranteed when the carrier space is constructed from the orbit of the group through a vacuum state $|\,0\rangle$. The representation becomes a maximum-weight one in which the annihilation operators are the adjoint[10] versions of the right-invariant partners of the vector fields in the polarization. In parallel to (19) we define: $\hat{L}^I_n \equiv i(\tilde{X}^I_{l^n})^R$ and $\hat{L}^{Sug}_n \equiv i(\tilde{X}^{Sug}_{l^n})^R$. Thus, the ultimate outcome of the formal polarization process we have just described, is to provide the form of the vectors in the representation space:

$$|\,\Psi\rangle = \prod_{i\in\{1,2,3\}} \hat{G}^i_{n_1}...\hat{G}^i_{n_i}\hat{L}^I_{p_1}...\hat{L}^I_{p_j}\;|\,0\rangle \quad (n_i \leq -1 \; if \; i \in \{1,2\}, n_3 \leq 0, p_j \leq -2) \quad , \tag{25}$$

where $|\,0\rangle$ is the vacuum state.

3.2 Unitarity

Now we shall consider the problem of unitarity. Therefore, we must introduce a notion of scalar product in the representation space. The standard way to implement this in GAQ is by using a measure on the group, which is explicitly computed from the group volume. This is a non-trivial issue for finite non-compact groups, although it can eventually be managed. However, in the case of infinite-dimensional groups, as is our case, we encounter a very hard problem. An alternative and consistent approach is to fix the norm of the vacuum state ($\langle 0\,|\,0\rangle = 1$) and then choose a rule for adjointness of the operators in the representation.

In order to decide the choice of adjoint operators, we impose consistency with the rigid $SU(1,1)$ subgroup, for which the scalar product can be introduced in a non-ambiguous way. Concretely, and due to the way the pseudoextension in the Kac-Moody subgroup has been made, the representation chosen for the rigid subgroup is associated with the discrete $SU(1,1) \approx SL(2,\mathbb{R})$ series. As can be seen from a explicit construction of this representation [9], we find:

$$\hat{Z}^\dagger = -\hat{Z}^* \; , \; \hat{\Phi}^\dagger = -\hat{\Phi} \quad , \tag{26}$$

[10] See next subsection.

which makes the $SU(1,1)$ representation unitary. These relations are translated to the Kac-Moody case in the following way:

$$(\hat{Z}_n)^\dagger = -\hat{Z}^*{}_{-n} \; , \; (\hat{\Phi}_n)^\dagger = -\hat{\Phi}_{-n} \quad . \tag{27}$$

This is therefore the rule for adjoint assignment [11] that fixes our scalar product.

Prior to the unitarity problem of the representation as regards the scalar product, another intimately related point must also be considered. Not all the algebra representations are lifted to group representations: there are globallity conditions. In our case, when we restrict ourselves to the subgroups generated by $\langle \tilde{X}^L_{z^q}, \tilde{X}^L_{z^{*q}}, \tilde{X}^L_{\phi^0}, \frac{\partial}{\partial\varphi}\rangle$, which form $SU(1,1)\tilde{\otimes}U(1)$, and consider the polarization given by $\langle \tilde{X}^L_{z^q}, \tilde{X}^L_{z^{*q}}, \tilde{X}^L_{\phi^0}\rangle \cap \mathcal{P}^{\mathcal{S}}_{\mathcal{KM}} = \langle \tilde{X}^L_{z^q}, \tilde{X}^L_{\phi^0}\rangle$, the global exponentiability of the wave functions forces the value of the pseudoextension[12], $\frac{K}{2} + \alpha n$, to satisfy the following inequalities:

$$\begin{array}{lll} n \leq 0 & , & \dfrac{K}{2} + \alpha n \leq -\dfrac{1}{2} \\ n > 0 & , & \dfrac{K}{2} + \alpha n \geq \dfrac{1}{2} \quad , \end{array} \tag{28}$$

which imply the $SU(1,1)$-Kac-Moody *fusion rule*,

$$\alpha \geq 0 \quad ; \quad -\frac{1}{2} \geq \frac{K}{2} \geq \; -\frac{1}{2}(2\alpha - 1) \, . \tag{29}$$

Thus, in our scheme, globallity fixes the sign of the central extension parameter α (which would have the opposite sign in the $SU(2)$ case, where the inequalities reverse sign) and provides a *natural* restriction on the permitted values of K according to the actual value of α. This second condition is analogous to the fusion rule in $SU(2)$ of Kac-Moody. For a general analysis of globallity conditions for central extensions of compact Kac-Moody groups we refer to [10] (see also [11]).

Now we return to the question of unitarity. If we consider the linear combinations:

$$\begin{aligned} \hat{J}^1_n &= \frac{1}{2}(\hat{Z}_n + \hat{Z}^*{}_n) \;\; , \;\; (\hat{J}^1_n)^\dagger = -\hat{J}^1_{-n} \\ \hat{J}^2_n &= \frac{1}{2}(\hat{Z}_n - \hat{Z}^*{}_n) \;\; , \;\; (\hat{J}^2_n)^\dagger = \hat{J}^2_{-n} \\ \hat{J}^3_n &= \hat{\Phi}_n \;\; , \;\; (\hat{J}^3_n)^\dagger = -\hat{J}^3_{-n} \;\; , \end{aligned} \tag{30}$$

and their commutators:

$$\begin{aligned} \left[\hat{J}^1_n, \hat{J}^1_m\right] &= -\alpha n \delta_{n+m}\hat{I} \\ \left[\hat{J}^2_n, \hat{J}^2_m\right] &= \alpha n \delta_{n+m}\hat{I} \\ \left[\hat{J}^3_n, \hat{J}^3_m\right] &= \alpha n \delta_{n+m}\hat{I} \; , \end{aligned} \tag{31}$$

[11] For the Virasoro modes, as in [1], we impose $(\hat{L}_n)^\dagger = \hat{L}_{-n}$.

[12] Which is directly linked to the Bargmann index of the representation.

then taking into account the positive sign of α, we find that $\hat{J}_n^1$ and $\hat{J}_n^2$ generate states of positive norm, whereas the states generated by $\hat{J}_n^3$ have a negative one. What we have found is that consistency with rigid $SU(1,1)$ representation theory enforces the existence of negative-norm states in the model, which spoil the unitarity of the theory. The presence of these states in non-compact Kac-Moody groups is a well-known feature (see [12, 13, 15, 16] and references therein).

According to the standard viewpoint, these states must be eliminated from the Hilbert space in order to find the *physical* quantum phase space. In the case of the bosonic string one encounters the same situation, but there the world-sheet reparametrization invariance compels the (Sugawara) Virasoro modes to act trivially, and this constraint eliminates these states. In our case, we cannot resort to the gauge invariance of a Lagrangian in order to motivate such a constraint. All we can adduce in our approach is mathematical consistency. In this line, good candidates for constraints (as indicated in [13]) are the Kac-Moody operators related to the Casimir of the rigid group. In the case of $SU(1,1)$, the only such possibility is the set of Virasoro operators. Therefore, our proposal here is to use the Sugawara-Virasoro operators in order to eliminate the non-physical states [13]. This is a well studied problem in the context of a bosonic string propagating on a curved spacetime. The answer is that Virasoro constraints are not enough to eliminate these vectors in the case of $SU(1,1)(\approx SL(2,\mathbb{R})$-Kac-Moody (there is not a no-ghost theorem in this case).

In [12] a number of solutions are proposed and finally discarded. The first proposed solution is to limit the possible rigid $SL(2,\mathbb{R})$ representations present in the Kac-Moody one by imposing the index of the $SL(2,\mathbb{R})$ (our K) to be bounded by the central extension parameter α. This is precisely our second condition in (29). This possibility is ruled out in [12] because it is not a natural condition for a string model, since at the quantum level it eliminates some excitations present in the classical theory. But we are not working with a string theory and therefore there is no reason for excluding this condition. Even more, this condition is necessary for lifting the Kac-Moody algebra to the group level. Thus in our construction, is globallity that is responsible for the elimination of negative-norm states.

We can summarize the previous considerations concerning the constraints on the Hilbert space that eliminate the negative-norm states [14] by writing:

$$\hat{L}_n^{Sug} \mid\Psi\rangle \;=\; 0 \;\;, n \leq 0 \quad \hat{L}_n^{Sug} \mid\Psi\rangle \sim \mid\Psi\rangle \;\;, n > 0$$

[13] A quite different and less harmful alternative treatment of the unitarity problem in non-compact infinite-dimensional groups is under study [14].

[14] An important subtlety of the $\hat{L}_0^{Sug}$ constraint, is that it forces an excited Kac-Moody state of level K to be constructed from a vacuum state with a very concrete non-trivial value of the Casimir of the rigid $SU(1,1)$ (see [12]). In order to maintain the possibility of different excitation levels for the matter modes as well as the notion of a true vacuum of the theory, we are obliged to consider a direct sum of irreducible representations constructed from the true vacuum $|0\rangle$ and mixed by *external* operators which play the same role as the position operators in string theory generating translations in the momentum space. We shall not dwell on any more on this point, since the description of the Kac-Moody Hilbert space to that degree of accuracy is not crucial for us at this stage.

$$-\frac{1}{2} \quad \geq \quad \frac{K}{2} \geq -\frac{1}{2}(2\alpha - 1) \quad , \tag{32}$$

where $\sim$ indicates that the two states must be identified by taking the quotient ($\hat{L}_n^{Sug} \mid \Psi\rangle, n > 0$ is a spurious state). The $\hat{L}_0^{Sug}$ constraint, together with the bound on K, establish an upper limit to the possible excited states that can appear in the theory (in fact, this was the reason in [12] for rejecting this representation).

3.3 Inönü-Wigner contraction

There is a mathematical construction devised by Inönü and Wigner [17] which allows the derivation of non-semisimple algebras and their representations from the case of semisimple ones by means of a contraction procedure. For the case of affine Lie algebras this contraction has been considered in [18] (see also [19]). The effect of this contraction can be seen as a decrease in the grade of non-linearity of certain contracted modes. Thus, the dynamics of these modes becomes more linear, which can be physically considered as a softening of the associated interaction. This mechanism can be of some interest when discussing the classical limit.

Let us briefly recall the fundamentals of the Inönü-Wigner contraction (for further details see [17, 18]). For the contraction to be possible, the original algebra must admit a decomposition:

$$\mathcal{G} = V_1 \oplus V_2 \tag{33}$$

such that if X^α, $\alpha = 1, ..., dimV_1$ and X^i, $i = 1, ..., dimV_2$ constitutes a basis for $\mathcal{G}$, then the Lie-algebra structure constants $C^{\alpha\beta}{}_i$ must vanish. In this case, we can contract with respect to V_1, by redefining the generators in V_2 with a multiplicative parameter λ that we make tend to zero ($\lambda \to 0$). After this limit, the V_1 subalgebra remains unaltered, but the contracted V_2 generators undergo a *linearization*. The structure of the resulting algebra is:

$$\begin{aligned} \left[X^\alpha, X^\beta\right] &= C^{\alpha\beta}{}_\gamma X^\gamma \\ \left[X^\alpha, X^i\right] &= C^{\alpha i}{}_j X^j \\ \left[X^i, X^j\right] &= 0 \quad . \end{aligned} \tag{34}$$

In our case, there indeed exists such a V_1 subalgebra. It is generated by the operators associated with the Cartan rigid subalgebra and the Virasoro modes. Contracting with respect to this subalgebra we find:

$$\begin{aligned} [\hat{Z}_n, \hat{Z}^*{}_m] &= 0 \\ \left[\hat{\Phi}_n, \hat{Z}_m\right] &= i\hat{Z}_{n+m} \ , \ \left[\hat{\Phi}_n, \hat{Z}^*{}_m\right] = -i\hat{Z}^*{}_{n+m} \\ \left[\hat{\Phi}_n, \hat{\Phi}_m\right] &= \alpha n \delta_{n+m} \hat{I} \\ \left[\hat{L}_n, \hat{Z}_m\right] &= -m\hat{Z}_{n+m} \ , \ \left[\hat{L}_n, \hat{Z}^*{}_m\right] = -m\hat{Z}^*{}_{n+m} \end{aligned} \tag{35}$$

$$
\begin{aligned}
\left[\hat{L}_n, \hat{\Phi}_m\right] &= -m\hat{\Phi}_{n+m} + im\frac{K}{2}\delta_{n+m}\hat{I} \\
\left[\hat{L}_n, \hat{L}_m\right] &= (n-m)\hat{L}_{n+m} + \frac{1}{12}(cn^3 - c'n)\delta_{n+m}\hat{I} \quad .
\end{aligned}
$$

Then we find that, in this limit, the z and z^* modes lose their dynamical character, and the physical degrees of freedom related to them disappear. On the other hand, as seen in the previous subsection, the modes associated with the Cartan generator are unphysical, so the only physical degrees of freedom after the contraction are the Virasoro ones.

It is very important to note that if the Virasoro modes were not present in the model, a dynamical content could be associated with the contracted modes as a trace of the pseudoextension K. In that case, after the Inönü-Wigner contraction, the Kac-Moody central extension would disappear for this modes, but the pseudo-extension would became a real extension (the same phenomenon happens for the Lorentz and Galileo groups):

$$
[\hat{Z}_n, \hat{Z}^*{}_m] = \kappa\delta_{n+m}\hat{I} \quad . \tag{36}
$$

However, the presence of the Virasoro generators in the physical algebra makes this extension impossible: it is forbidden by the Jacobi identities. What we have found is that the presence of *gravity* modes imposes constraints to the size of physical phase space.

We shall comment further on the role of this Inönü-Wigner contraction when we consider the physical interpretation of the model.

4 Physical analysis of the model

In this section we study the physical content of the mathematical objects presented in the previous sections.

As far as the quantum model is concerned, our chief aim is to give an interpretation to the states in the Hilbert space. We first focus on the states of the form:

$$
\hat{L}^I_{n_1}...\hat{L}^I_{n_j}\,|0\rangle \quad n_1,...,n_j \leq -2 \quad , \tag{37}
$$

generated by the intrinsic Virasoro modes. The role of these vectors is that of generating the underlying spacetime structure, exactly in the same way the Virasoro modes work in [1] (see this reference for further details). They span a Virasoro irreducible representation with $c^I (= c - c^{Sug}) = c' > 1$, which is reduced under its kinematical $sl(2,\mathbb{R})$ subalgebra, producing an ensemble of $sl(2,\mathbb{R})$ irreducible representations, denoted by $R^{(N)}$. An AdS spacetime of *radius* $R = \frac{c^I}{\sqrt{N(N-1)}}$ is associated with each $R^{(N)}$ representation, while a specific vector state $|N,n,i\rangle$ in $R^{(N)}$ represents a particular state of that spacetime (n is an excitation index and i refers to the degeneration of $R^{(N)}$). These $|N,n,i\rangle$ states constitute an orthogonal basis of the Virasoro representation. The interpretation of the intrinsic Virasoro modes is, therefore, the analogue of the Virasoro modes in the pure gravity model, being the basis of the spacetime skeleton.

The novel feature is the presence of the *matter* degrees of freedom. A general state can be formally written as:

$$|\,\Psi\rangle = \sum_{N,n,i} (Physical\ Kac - Moody\ modes)\,|\,N,n,i\rangle \quad . \tag{38}$$

We may think of the states generated by the physical modes $\hat{J}^1_n$ and $\hat{J}^2_n$ (or equivalently by $\hat{Z}_n$ and $\hat{Z}^*{}_n$) as being the quanta of some *quantum fields* representing matter. Of course, these are not fields in the ordinary sense, since we lack of a unique spacetime background (we have a whole ensemble of them) on which these objects would have support. Only when we consider a state completely lying on a specific spacetime (that is, when N and i are fixed in the previous sum), does a standard notion of field show up, and then $\hat{J}^1_n$ and $\hat{J}^2_n$ can be seen as *excitations* of them. In general, a physical state is a linear superposition of different spacetimes, each one supporting a different content of matter excitation modes. Therefore, matter has an essential global and non-local character.

A very important point is the fact that matter degrees of freedom are not free modes, as a consequence of their non-trivial commutation relations. Thus, if a state with a given matter field content suffers the action of a (matter) perturbation, the reordering process (originated when passing the lowering operator to the right until it annihilates the vacuum) causes a mixing among the matter fields which results in a change in the distribution of matter modes. This makes the structure of the Hilbert space a very complicated one, a situation that gets even worse when the Sugawara constraints (a part of the Virasoro gravity modes) are taken into account. Despite the complexity of the quantum phase space, the physical image of the system is quite simple: we have two interacting quantum matter fields spread over different spacetimes.

The effect of gravity is the consequence of the action of the complete Virasoro modes: $\hat{L}_n = \hat{L}^{Sug}_n + \hat{L}^I_n$. When one of these modes acts on a given physical state $|\,\Psi\rangle$, it has a double effect. On the one hand, and due to the $\hat{L}^{Sug}_n$ $(n > 0)$ part, it affects the matter distribution (again as a consequence of non-trivial commutation) thus creating a gravity-matter interaction, and on the other hand, the $\hat{L}^I_n$ changes the spacetime distribution of *Universe*, in the sense of [1]. This double effect enriches the dynamics of the model with respect to the pure gravity case. We see that the *intrinsic* part of Virasoro $(\hat{L}^I_n)$ is responsible for the spacetime notion and its dynamics, whereas the *orbital* part $(\hat{L}^{Sug}_n)$, absent when no matter is present, is the responsible of gravity-matter interaction.

The semi-classical limit is accomplished by making $c \to \infty$. In this limit, the different spacetimes collapse into a unique AdS with a very large radius. The matter modes are therefore defined on the same spacetime support, giving rise to an interpretation as excitations of standard matter fields.

The dynamics of these fields can be studied by using the semiclassical formalism presented in **section 2**. We note that the quantum condition for the existence of a spacetime notion, $c - c^{Sug} = c'$, becomes indistinguishable from the classical condition $c = c' - 3\frac{K^2}{\alpha}$ in this limit, since c^{Sug} and $\frac{K^2}{4\alpha}$ are upper-bounded quantities, so that both conditions are consistent. As we saw in detail in that section, a semiclassical action can be constructed for the classical fields. This action is the sum of a gravity, a matter and

an interaction term. The matter term has a perturbative expression which at the lowest order represents three free scalar fields. This can be seen directly from the form of Θ^{KM} (7). The higher-order correction terms couple these scalar fields, producing non-trivial matter dynamics. The non-physical field F_ϕ can be seen as an auxiliary field necessary for implementing the matter dynamics. The gravity term has exactly the same form we found in [1], and we take from there the interpretation for the field F_l, as an effective classical metric field giving rise to a dynamical correction to the kinematical background metric:

$$ds^2 = ds^2_{AdS(R\gg 1)} \ + \ \partial_\lambda F_l d\lambda du \ . \tag{39}$$

The presence of the gravity-matter interaction term shows the influence of matter in the classical metric notion, as desired.

The meaning of the physical matter fields, F_z and F_{z^*} is not quite clear. A possibility is that they are simply classical non-free scalar fields in interaction with gravity, but this then poses the problem of identifying the kind of physical matter or interaction they correspond to. But another possibility is that these degrees of freedom only makes sense in a strong interaction regime, in such a way that in the soft classical limit they decouple. The mathematical justification for this option is the possibility of incorporating an Inönü-Wigner contraction in the model, in which case the physical matter modes would appear corrected by a multiplicative parameter λ. In the weak interaction limit, $\lambda \to 0$, the modes associated with z^n and z^{*n} lose their dynamical content. This is a non-trivial fact due to the presence of gravity modes, and manifests itself by a decrease in the size of the classical phase space with respect to the quantum one. We have *fields* which possess a physical existence only in the non-linear strong regime, and become trivial in the weak one. A mechanism like this could be interesting in the study of the softening of gravity singularities: it would indicate the existence of *exotic* physics in the surroundings of singularities which disappear as we move away from them and with no analogues in the classical theory.

One potential snag in the previous discussion is the specification of the nature of the parameter λ . A number of possibilities can be found inside the model by combining the constants α, c and c'. Unfortunately the model is not sufficiently predictive so as to select a specific one.

5 Conclusions

We have tried to formulate the dynamics of gravity in the presence of matter, assuming the very strong hypothesis that the whole physical content of the system is encoded in abstract symmetry principles. On this line, we have constructed a quantum model whose physical states represent linear superpositions of AdS spacetimes with different radii and where matter excitations have a definite non-local character; in the general case, they have support on various spacetimes simultaneously. However, a notion for the probability of a matter excitation to lie on a specific spacetime makes sense due to the orthogonality of

spacetime vectors and the trivial commutation of matter and intrinsic gravity modes. It should be remarked that the quantum realization of spacetime is directly associated with the $SL(2,\mathbb{R})$-anomaly, eventually solved by means of a higher-order polarization closing a proper algebra. (In general, higher-order polarizations only have to close weakly, i.e. on solutions.) This allows a well-defined integral manifold supporting the reduced wave functions.

Global features of the symmetry structure have proven to be crucial for the consistency of the model. They have allowed us to eliminate the *ad hoc* character of the crucial restrictions (29), by deducing the natural and unavoidable necessity of them [15]. In particular, these restrictions have endowed the matter degrees of freedom with a peculiar behaviour in the quantum regime, since it limits the excitation capabilities of these modes.

We have found the phenomenon of an enlarging of the physical phase space at the quantum level; that is, the quantum emergence of physical degrees of freedom which are absent from the classical limit. This can be seen both in the quantum acquisition of dynamical content of the diffeomorphisms and in the classical loss of physical presence of the matter modes via an Inönü-Wigner contraction process. This is an intrinsically interesting question which probably survives this particular model and could be present in completely different physical systems.

Finally, we recognise the essential limitations and difficulties of taking the symmetry hypothesis to its ultimate consequences in the concrete formulation of this model. Even though it has an intrinsic beauty and power, it poses ambiguity problems related to the physical interpretation of the constructed objects. It does not seem to be strong enough to fix a unique possibility among the different choices it raises. However, we argue that this approach provides profound physical insight when inserted in a more general framework. In the context of gravity, the inclusion of genuinely metric notions should enrich the physical system by suggesting new solutions, so the extension of this approach to higher dimensions is an urgent necessity.

6 Acknowledgements

We thank J. Guerrero for very useful and illuminating discussions.

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